On a family of operator means involving the power difference means (Theory of operator means and related topics)

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CITATION:

宇田川, 陽一 ...[et al]. On a family of operator means involving the power difference means (Theory of operator means and related topics). 数理解析研究所講究録 2015, 1935: 45-58: KJ00009772905.

ISSUE DATE:

2015-04

数理解析研究所講究録
第 1935 巻 2015 年 45-58

On a family of operator means involving the power difference means

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Abstract

It is so famous that power mean is interpolational mean which interpolates arithmetic, geometric, and harmonic means. Power difference mean and stolarsky mean are known as the interpolational means too. Moreover, these all means are operator means.

In this report, we introduce a new way to get a family of operator means and obtain a new type of interpolational mean $F_{p,q}(t)$. This interpolational mean $F_{p,q}(t)$ interpolates power mean, power difference mean, and a part of stolarsky mean.

1 Introduction

In this report, we study operator mean and operator monotone function. First of all, we introduce some symbols and definitions used in this paper.

Let $\mathcal{H}$ be a complex Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, and $\mathcal{B}(\mathcal{H})$ be a set of all bouded linear operators on $\mathcal{H}$. An operator $A\in\mathcal{B}(\mathcal{H})$ is said to be positive if and only if $\langle Ax,x\rangle\geq 0$ for all $x\in\mathcal{H}$. We denote a positive operator A by $A\geq 0$. Let $\mathcal{B}(\mathcal{H})_{+}$ be a set of all positive operators in $\mathcal{B}(\mathcal{H})$. For self-adjoint operators $A,B\in\mathcal{B}(\mathcal{H})$, $A\leq B$ means $B-A$ is positive.

A function $f(t)$ defined on an interval I in $\mathbb{R}$ is called an *operator monotone function*, provided $A\leq B$ implies $f(A)\leq f(B)$ for every pair A and B whose spectra $\sigma(A)$ and $\sigma(B)$ lie in I. The function $f(t)=t^{\alpha}$ $(0\leq\alpha\leq 1)$ is a well-known operator monotone function. By this fact, we get

$$0\leq A\leq B\Longrightarrow\frac{A^{\alpha}-I}{\alpha}\leq\frac{B^{\alpha}-I}{\alpha}.$$

So we can find $f(t)=\log t$ is an operator monotone function by taking limit $\alpha\downarrow 0$.

The map $\sigma : \mathcal{B}(\mathcal{H})_+^2 \to \mathcal{B}(\mathcal{H})_+$ is called an *operator mean*[6] if the operator $A\sigma B$ satisfies the following four conditions for $A, B \in \mathcal{B}(\mathcal{H})_+$;

(1) $A \leq C$ and $B \leq D$ implies $A\sigma B \leq C\sigma D$,

(2) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$ for all self-adjoint $C \in \mathcal{B}(\mathcal{H})$,

(3) $A_n \searrow A$ and $B_n \searrow B$ imply $A_n\sigma B_n \searrow A\sigma B$,

(4) $I\sigma I = I$.

Next theorem is so important to study operator means;

Theorem 1 (Kubo-Ando[6]). *For any operator mean σ, there uniquely exists an operator monotone function $f \geq 0$ on $[0, \infty)$ with $f(1) = 1$ such that*

$$f(t)I = I\sigma(tI), \quad t \geq 0.$$

Then the following hold:

(1) *The map $\sigma \mapsto f$ is a one-to-one onto affine mapping from a set of all operator means to a set of all non-negative operator monotone functions on $[0, \infty)$ with $f(1) = 1$. Moreover, $\sigma \mapsto f$ preserves the order. Therefore, when $\sigma_i \mapsto f_i (i = 1, 2)$,*

$$A\sigma_1 B \leq A\sigma_2 B \quad (A, B \in \mathcal{B}(\mathcal{H})_+) \Longleftrightarrow f_1(t) \leq f_2(t) \quad (t \geq 0).$$

(2) *When $A > 0$,* $\quad A\sigma B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}.$

By previous theorem, it is enough to think about operator monotone function when we think about operator means. So we will only think about "positive" operator monotone functions on $[0, \infty)$ in the following.

Next we introduce some types of operator means;

Definition 1. *Let σ be an operator mean and $f(t)$ be a corresponding function of σ.*

(1) *The operator mean characterized by $\frac{t}{f(t)}$ is called dual of σ,*

(2) *The operator mean characterized by $f(t^{-1})^{-1}$ is called adjoint of σ,*

(3) *The operator mean characterized by $tf(t^{-1})$ is called orthogonal of σ,*

(4) *If $f(t)$ satisfies $f(t) = tf(t^{-1})$, then σ is called a symmetric operator mean.*

If σ is symmetric, then $A\sigma B = B\sigma A$. Lastly, we introduce some examples of operator means;

Example 1. *Arithmetic Mean:* $f(t) = \frac{1+t}{2}$

$$A \nabla B = A^{\frac{1}{2}} \left(\frac{1}{2} \left(I + A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \right) \right) A^{\frac{1}{2}} = \frac{A+B}{2}.$$

Logarithmic Mean:

$$A \lambda B = A^{\frac{1}{2}} f \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \right) A^{\frac{1}{2}}, \quad f(t) = \frac{t-1}{\log t}.$$

Geometric Mean: $f(t) = t^{\frac{1}{2}}$

$$A \sharp B = A^{\frac{1}{2}} \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \right)^{\frac{1}{2}} A^{\frac{1}{2}}.$$

Harmonic Mean: $f(t) = \frac{2t}{1+t} = 2\left(1+t^{-1}\right)^{-1}$

$$A ! B = A^{\frac{1}{2}} \left\{ 2 \left(I + \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \right)^{-1} \right)^{-1} \right\} A^{\frac{1}{2}} = 2\left(A^{-1} + B^{-1}\right)^{-1}.$$

2 Examples of Interpolational Means

Throughout this paper, if an operator mean interpolates some operator means, then we call it "interpolational mean". We describe some examples of interpolational means in the following;

Example 2 (Power mean[5]). *Let* $-1 \le s \le 1$. *Then*

$$P_s(t) = \left(\frac{1+t^s}{2} \right)^{\frac{1}{s}}$$

is an operator monotone function of $t > 0$, *and* $P_{s_1}(t) \le P_{s_2}(t)$ *holds for* $-1 \le s_1 \le s_2 \le 1$. *(The case* $s = 0$ *is defined as the limit.)*

$s = 1$ *(Arithmetic Mean)* :

$$P_1(t) = \frac{1+t}{2}.$$

$s \to 0$ *(Geometric Mean)* :

$$P_0(t) := \lim_{s \to 0} P_s(t) = t^{\frac{1}{2}}.$$

$s = -1$ *(Harmonic Mean)* :

$$P_{-1}(t) = \left(\frac{1+t^{-1}}{2} \right)^{-1} = \frac{2t}{1+t}.$$

Example 3 (Stolarsky Mean[7]). *Let* $-2 \leq s \leq 2$. *Then,*

$$ST_s(t) = \left\{ \frac{t^s - 1}{s(t-1)} \right\}^{\frac{1}{s-1}}$$

is an operator monotome function of $t > 0$. (*The case* $s = 0, 1$ *are defined as the limits.*)

$s = 2$ (*Arithmetic Mean*) :

$$ST_2(t) = \frac{1+t}{2}.$$

$s \to 1$ (*Identric Mean*) :

$$ST_1(t) := \lim_{s \to 1} ST_s(t) = \exp\left\{ \frac{t \log t}{t-1} - 1 \right\}.$$

$s \to 0$ (*Logarithmic Mean*) :

$$ST_0(t) := \lim_{s \to 0} ST_s(t) = \frac{t-1}{\log t}.$$

$s = -1$ (*Geometric Mean*) :

$$ST_{-1}(t) = t^{\frac{1}{2}}.$$

Example 4 (Power Difference Mean[3, 4, 5]). *Let* $-1 \leq r \leq 2$. *Then*

$$g_r(t) = \frac{(r-1)(t^r - 1)}{r(t^{r-1} - 1)}$$

is an operator monotone function of $t > 0$ *and* $g_{r_1}(t) \leq g_{r_2}(t)$ *holds for* $-1 \leq r_1 \leq r_2 \leq 2$.

Due to the following relation, we treat power difference mean as $PD_s(t)$ *in this report* ;

$$g_r(t) = \frac{(r-1)(t^r - 1)}{r(t^{r-1} - 1)} \; (-1 \leq r \leq 2) \iff PD_s(t) = \frac{s(t^{1+s} - 1)}{(1+s)(t^s - 1)} \; (-2 \leq s \leq 1)$$

(*The case* $s = 0, 1$ *are defined as the limits.*)

$s = 1$ (*Arithmetic Mean*) :

$$PD_1(t) = \frac{1+t}{2}.$$

$s \to 0$ (*Logarithmic Mean*) :

$$PD_0(t) := \lim_{s\to 0} PD_s(t) = \frac{t-1}{\log t}.$$

$s = \frac{-1}{2}$ (*Geometric Mean*) :

$$PD_{\frac{-1}{2}}(t) = t^{\frac{1}{2}}.$$

$s \to -1$ (*Adjoint of Logarithmic Mean*) :

$$PD_{-1}(t) := \lim_{s\to -1} PD_s(t) = \frac{t\log t}{t-1} = PD_0(t^{-1})^{-1}.$$

$s = -2$ (*Harmonic Mean*) :

$$PD_{-2}(t) = \frac{2t}{1+t} = PD_1(t^{-1})^{-1}.$$

Operator monotonicity of power difference mean is delivered by the following integration. Let $f_s(\alpha;t) = ((1-\alpha)+\alpha t^s)^{\frac{1}{s}}$ $(-1 \le s \le 1)$. Then

$$\left(F_s(t) =\right) \int_0^1 f_s(\alpha;t)d\alpha = \left[\frac{1}{t^s-1}\frac{s}{s+1}(\alpha(t^s-1)+1)^{\frac{1}{s}+1}\right]_0^1$$
$$= \frac{s(t^{1+s}-1)}{(1+s)(t^s-1)}$$

is an operator monotone function of $t \ge 0$ and $F_{s_1}(t) \le F_{s_2}(t)$ holds for $-1 \le s_1 \le s_2 \le 1$.

This derivation is so beautiful, but has one probrem. Power Difference Mean is an operator monotone function for $-2 \le s \le 1$. However, we can not show operator monotonicity of $F_s(t)$ for $-2 \le s < -1$ by using this technique.

3 Main Results

In this section, we introduce a new way to get a family of operator means and obtain a new interpolational mean by applying it.

For a natural number k, let $u(t)$ be a positive function on $[0,\infty)$ defined by

$$u(t) := r\prod_{i=1}^{k}(t+a_i)^{p_i},\ (0 \le a = a_1 < a_2 < \cdots < a_k = b, 1 \le p_1, 0 < p_i, 0 < r).$$

We remark that M.Uchiyama[9] has shown $u^{-1}(t)$ is an operator monotone function.

Theorem 2. *Let μ be a probability measure on $[0,1]$ and $\{f(\alpha;t)|\alpha \in [0,1]\}$ be a family of positive valued operater monotone functions of $t \geq 0$. Assume for each $t \geq 0$, the map $\alpha \mapsto f(\alpha;t)$ is continuous. Then*

$$F(t) = u\left(\int_0^1 u^{-1}\left(f(\alpha;t)\right) d\mu(\alpha) + b - a\right)$$

is an operator monotone function.

Proof of Theorem 2. As every operator monotone function $f \geq 0$ on $[0,\infty)$ is a Pick function, it is enough to show that $u\left(\sum_j \beta_j u^{-1}(f(\alpha_j;t)) + b - a\right)$ is a Pick function for any positive numbers $\beta_1, \ldots, \beta_m$ which satisfy $\sum_j \beta_j = 1$. From the assumption, $u^{-1}\left(f(\alpha;t)\right)$ is an operator monotone function by the above attention.

For a complex number z which is in the upper half complex plane,

$$\begin{aligned}
0 &< \arg\left(u\left(\sum_j \beta_j u^{-1}(f(\alpha_j;z)) + b - a\right)\right) \\
&= \sum_i p_i \arg\left(\sum_j \beta_j u^{-1}(f(\alpha_j;z)) + b - a + a_i\right) \\
&\le \sum_i p_i \arg\left(\sum_j \beta_j u^{-1}(f(\alpha_j;z)) + b\right) \\
&= \sum_i p_i \arg\left(\sum_j \beta_j \left(u^{-1}(f(\alpha_j;z)) + b\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_i p_i \arg\left(u^{-1}(f(\alpha_{j_0};z)) + b\right) \\
&\le \sum_i p_i \arg\left(u^{-1}(f(\alpha_{j_0};z)) + a_i\right) \\
&= \arg(u(u^{-1}(f(\alpha_{j_0};z)))) \\
&= \arg f(\alpha_{j_0};z) < \pi.
\end{aligned}
$$

□

Corollary 1. *Let $\{f(\alpha;t)|\alpha \in [0,1]\}$ be a family of positive valued operator monotone functions of $t \ge 0$. Assume for each $t \ge 0$, the map $\alpha \mapsto f(\alpha;t)$ is continuous. Then for each $p \in [-1,1] \setminus \{0\}$,*

$$
F_p(t) := \left(\int_0^1 f(\alpha;t)^p d\alpha\right)^{\frac{1}{p}}
$$

is an operator monotone function of $t \ge 0$. Moreover, for each fixed $t \ge 0$, $F_{p_1}(t) \le F_{p_2}(t)$ holds for $p_1, p_2 \in [-1,1] \setminus \{0\}$, $p_1 \le p_2$.

Corollary 1 is a special case of Theorem 1 by taking $u(t) = t^{\frac{1}{p}}$. It is also obtained by using Thompson metric and Banach fixed point theorem. We now give another proof of Collorary 1. Firstly, we prepare the definition and properties of Thompson metric to prove Corollary 1. In what follows, let $\mathbb{P}$ be a cone of strictry positive operators.

Definition 2. *For $A, B \in \mathbb{P}$, let*

$$
M(B/A) = \inf\{\alpha > 0; B \le \alpha A\}.
$$

Then the Thompson metric is defined by

$$
d_\infty(A,B) = \max\{\log M(B/A), \log M(A/B)\}.
$$

Thompson metric is a complete metric on $\mathbb{P}$. (Thompson 1963[8]) Thompson metric have nice two properties[1][2];

(1) Let $A, B \in \mathbb{P}$ and $d_\infty(A,B) = \log m$. Then,

$$
m^{-1}B \le A \le mB.
$$

Conversely, if m satisfies $m^{-1}B \le A \le mB$, then $d_\infty(A,B) \le \log m$.

(2) Let $A, B, C, D \in \mathbb{P}$. Then, for any $\alpha \in [0,1]$,

$$
d_\infty(A\sharp_\alpha B, C\sharp_\alpha D) \le (1-\alpha)d_\infty(A,C) + \alpha d_\infty(B,D)
$$

where $A\sharp_\alpha B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\alpha}A^{\frac{1}{2}}$.

Lemma 1. *Let d_∞ be a Thompson metric of $\mathbb{P}$. Then, for $X, Y, A > 0$ and $s \in (0,1]$,*

$$d_\infty\left(\int_0^1 (X\sharp_s f(\alpha; A))d\alpha, \int_0^1 (Y\sharp_s f(\alpha; A))d\alpha\right)$$

$$\leq \sup_{\alpha\in[0,1]} d_\infty\Big(X\sharp_s f(\alpha; A), Y\sharp_s f(\alpha; A)\Big).$$

Proof of Lemma 1. Let

$$\sup_{\alpha\in[0,1]} d_\infty\Big(X\sharp_s f(\alpha; A), Y\sharp_s f(\alpha; A)\Big) = \log m.$$

From property (1),

$$m^{-1}(Y\sharp_s f(\alpha; A)) \leq X\sharp_s f(\alpha; A) \leq m(Y\sharp_s f(\alpha; A))$$

holds for any $\alpha \in [0,1]$. Therefore,

$$m^{-1}\int_0^1 (Y\sharp_s f(\alpha; A))d\alpha \leq \int_0^1 (X\sharp_s f(\alpha; A))d\alpha \leq m\int_0^1 (Y\sharp_s f(\alpha; A))d\alpha,$$

namely,

$$d_\infty\left(\int_0^1 (X\sharp_s f(\alpha; A))d\alpha, \int_0^1 (Y\sharp_s f(\alpha; A))d\alpha\right)$$

$$\leq \log m = \sup_{\alpha\in[0,1]} d_\infty\Big(X\sharp_s f(\alpha; A), Y\sharp_s f(\alpha; A)\Big).$$

□

Lemma 2. *The map $F : \mathbb{P} \longrightarrow \mathbb{P}$ defined by*

$$F(X) = \int_0^1 (X\sharp_s f(\alpha; A))d\alpha \ \big(s \in (0,1]\big)$$

is a contractive map. Moreover, the following equation

$$X = \int_0^1 (X\sharp_s f(\alpha; A))d\alpha \ \big(s \in (0,1]\big)$$

has the unique positive solution on $\mathbb{P}$ and its solution coincides with

$$\left(\int_0^1 f(\alpha; A)^s d\alpha\right)^{\frac{1}{s}}.$$

Proof of Lemma 2. Let $X, Y \in \mathbb{P}$. Then,

$$\begin{aligned} d_\infty(F(X), F(Y)) &= d_\infty\left(\int_0^1 (X\natural_s f(\alpha; A))d\alpha, \int_0^1 (Y\natural_s f(\alpha; A))d\alpha\right) \\ &\le \sup_{\alpha\in[0,1]} d_\infty\Big(X\natural_s f(\alpha; A), Y\natural_s f(\alpha; A)\Big) \\ &\le \sup_{\alpha\in[0,1]} \{(1-s)d_\infty(X,Y) + sd_\infty(f(\alpha; A), f(\alpha; A))\} \\ &= (1-s)d_\infty(X,Y), \end{aligned}$$

where the first inequality holds from Lemma 1 and the second one holds from property (2). Since $1-s \in [0,1)$, F is a contractive map. From Banach fixed point theorem, F has the unique fixed point, namely,

$$X = F(X) = \int_0^1 (X\natural_s f(\alpha; A))d\alpha$$

has the unique positive solution on $\mathbb{P}$. Next, we show its sulution coincides with

$$X_0 := \left(\int_0^1 f(\alpha; A)^s d\alpha\right)^{\frac{1}{s}}$$

by substitution. Since $X_0\natural_s f(\alpha; A) = X_0^{1-s}f(\alpha; A)^s$ and

$$X_0 = \left(\int_0^1 f(\alpha; A)^s d\alpha\right)^{\frac{1}{s}} \Longrightarrow X_0^s = \int_0^1 f(\alpha; A)^s d\alpha,$$

$$\begin{aligned} \int_0^1 (X_0\natural_s f(\alpha; A))d\alpha &= \int_0^1 X_0^{1-s}f(\alpha; A)^s d\alpha = X_0^{1-s}\int_0^1 f(\alpha; A)^s d\alpha \\ &= X_0^{1-s}X_0^s = X_0. \end{aligned}$$

□

Proof of Collorary 1. Firstly, we prove operator monotonicity about the case $p \in (0,1]$. For $0 < A \le B$, let define the maps F, G as

$$F(X) = \int_0^1 (X\natural_p f(\alpha; A))d\alpha, \ G(X) = \int_0^1 (X\natural_p f(\alpha; B))d\alpha.$$

Then from Lemma 2, $F(X) = X$ and $G(X) = X$ have the following unique sulutions in $\mathbb{P}$ respectively:

$$X_0 = \left(\int_0^1 f(\alpha; A)^p d\alpha\right)^{\frac{1}{p}}, X_1 = \left(\int_0^1 f(\alpha; B)^p d\alpha\right)^{\frac{1}{p}}.$$

Moreover, $F^k(X) \leq G^k(X)$ holds for all $k \in \mathbb{N}$, where F^k is the k-times composite of F. For any $X > 0$, there exist $\lim_{k\to\infty} F^k(X)$, $\lim_{k\to\infty} G^k(X)$ and

$$\left(\int_0^1 f(\alpha; A)^p d\alpha\right)^{\frac{1}{p}} = X_0 = \lim_{k\to\infty} F^k(X),$$

$$\left(\int_0^1 f(\alpha; B)^p d\alpha\right)^{\frac{1}{p}} = X_1 = \lim_{k\to\infty} G^k(X)$$

by Banach fixed point theorem. Therefore, we have

$$\left(\int_0^1 f(\alpha; A)^p d\alpha\right)^{\frac{1}{p}} = \lim_{k\to\infty} F^k(X) \leq \lim_{k\to\infty} G^k(X) = \left(\int_0^1 f(\alpha; B)^p d\alpha\right)^{\frac{1}{p}}.$$

The case $p \in [-1, 0)$ is also obtained by replacing p into $-p$ and $X = X^{-1}$.

Next we show the monotonicity of $p \in [-1, 1] \setminus \{0\}$. For $0 < p_1 \leq p_2 \leq 1$, $t^{\frac{p_1}{p_2}}$ is a concave function. By Jensen's inequality, we have

$$\int_0^1 (f(\alpha; t)^{p_2})^{\frac{p_1}{p_2}} d\alpha \leq \left(\int_0^1 f(\alpha; t)^{p_2} d\alpha\right)^{\frac{p_1}{p_2}}$$

and find $F_{p_1}(t) \leq F_{p_2}(t)$ $(t > 0,\ 0 < p_1 \leq p_2 \leq 1)$. Likewise, we can get $F_{q_1}(t) \leq F_{q_2}(t)$ $(t > 0,\ -1 \leq q_1 \leq q_2 < 0)$. Moreover, we can find $F_{-p}(t) \leq F_p(t)$ $(t > 0,\ p \in (0, 1])$ from convexity of t^{-1} and Jensen's inequality. From the above,

$$F_p(t) \leq F_q(t) \quad (t > 0,\ p, q \in [-1, 1] \setminus \{0\},\ p \leq q).$$

□

In Corollary 1, we don't think about the case $p = 0$. The case $p = 0$ is defined as the limit if it exists. And then, we have the following equation by l'Hopital's rule;

$$F_0(t) := \lim_{p\to 0} F_p(t) = \exp\left(\int_0^1 \log f(\alpha; t) d\alpha\right).$$

4 Applications

In Section 3, we have had a new way to get a family of operator means. By applying it, we obtain a new family of operator means at the next theorem. After that, we show that it is an interpolational mean which interpolates power mean, power difference mean, and a part of stolarsky mean.

Theorem 3. *For $s \in [-1,1] \setminus \{0\}$ and $r \in [-1,1] \setminus \{0\}$,*

$$F_{s,r}(t) = \left(\int_0^1 (f_r(\alpha;t))^s \, d\alpha \right)^{\frac{1}{s}}$$

is an operator monotone function of $t \geq 0$ and

$$F_{s_1,r}(t) \leq F_{s_2,r}(t), \quad F_{s,r_1}(t) \leq F_{s,r_2}(t)$$

hold for $s_1, s_2 \in [-1,1] \setminus \{0\}, s_1 \leq s_2, r_1, r_2[-1,1] \setminus \{0\}, r_1 \leq r_2$.

Theorem 3 is obtained by putting $f_r(\alpha;t) = [(1-\alpha) + \alpha t^r]^{\frac{1}{r}}$ and $p = s$ in Corollary 1. In Theorem 2, the case $s, r = 0$ can be considered as the limits of $s, r \to 0$, respectively.

By simple computation, we get

$$F_{s,r}(t) = \left(\frac{r(t^{s+r} - 1)}{(s+r)(t^r - 1)} \right)^{\frac{1}{s}}.$$

We call $F_{s,r}(t)$ "extension of power difference mean".

Proposition 1. *Extension of power difference mean $F_{s,r}(t)$ is an interpolational mean which interpolates power mean, power difference mean, and a part of stolarsky mean.*

Proof of Proposition 1. By taking $s = r$, we get power mean

$$F_{s,s}(t) = \left(\frac{1+t^s}{2} \right)^{\frac{1}{s}} = P_s(t) \quad (-1 \leq s \leq 1).$$

Thus $F_{s,r}(t)$ interpolates power mean and the means which power mean interpolates.

Next, we show $F_{s,r}(t)$ interpolates power difference mean. By taking $s = 1$, we get

$$F_{1,r}(t) = \frac{r(t^{1+r} - 1)}{(r+1)(t^r - 1)} \quad (-1 \leq r \leq 1).$$

By taking $s = -1$ and thinking parameter range, we get

$$F_{-1,r}(t) = \frac{(r-1)(t^r - 1)}{r(t^{r-1} - 1)} (-1 \leq r \leq 1) \Longleftrightarrow \frac{p(t^{1+p} - 1)}{(p+1)(t^p - 1)} (-2 \leq p \leq 0)$$

Connecting both parameter range, we have power difference mean

$$PD_s(t) = \frac{s(t^{1+s} - 1)}{(1+s)(t^s - 1)} \quad (-2 \leq s \leq 1).$$

Lastly, we think about the case $s = p - 1$ and $r = 1$. Simple substitution derives

$$F_{p-1,1}(t) = \left\{ \frac{t^p - 1}{p(t-1)} \right\}^{\frac{1}{p-1}} = ST_p(t) \quad (0 \le p \le 2).$$

It follows from Theorem 3 that the above is an operator monotone function of $t \ge 0$ and $ST_{p_1}(t) \le ST_{p_2}(t)$ holds for $0 \le p_1 \le p_2 \le 2$. □

Remark 1.

$$ST_p(t) = \left\{ \frac{t^p - 1}{p(t-1)} \right\}^{\frac{1}{p-1}} \qquad (-2 \le p \le 2).$$

Stolarsky mean is an operator monotone function for $-2 \le p \le 2$. However, we can not show operator monotonicity for $-2 \le p < 0$ by using operator monotonicity of $F_{s,r}(t)$, directly.

At the end of the paper, we introduce a property of $F_{s,r}(t)$.

Proposition 2. *Let*

$$F_{s,r}(t) = \left(\frac{r(t^{s+r} - 1)}{(s+r)(t^r - 1)} \right)^{\frac{1}{s}}$$

and $\sigma_{s,r}$ be an operator mean which is characterized by $F_{s,r}(t)$, i.e., for $A, B > 0$,

$$A\sigma_{s,r}B = A^{\frac{1}{2}} F_{s,r}\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \right) A^{\frac{1}{2}}.$$

Then $\sigma_{s,r}$ is symmetric operator mean, and both of the dual and the adjoint of $\sigma_{s,r}$ coincide with $\sigma_{-s,-r}$.

Proof of Proposition 2.

$$\begin{aligned} tF_{s,r}(t^{-1}) &= t \left(\int_0^1 \{(1-\alpha) + \alpha t^{-r}\}^{\frac{s}{r}} d\alpha \right)^{\frac{1}{s}} \\ &= \left(\int_0^1 t^s \{(1-\alpha) + \alpha t^{-r}\}^{\frac{s}{r}} d\alpha \right)^{\frac{1}{s}} \\ &= \left(\int_0^1 \{(1-\alpha)t^r + \alpha\}^{\frac{s}{r}} d\alpha \right)^{\frac{1}{s}} = F_{s,r}(t). \end{aligned}$$

Hence, $F_{s,r}(t)$ is symmetric.

Similarly,

$$\frac{t}{F_{s,r}(t)} = t \left(\int_0^1 \{(1-\alpha) + \alpha t^r\}^{\frac{s}{r}} d\alpha \right)^{\frac{-1}{s}}$$

$$= \left(\int_0^1 t^{-s}\{(1-\alpha)+\alpha t^r\}^{\frac{s}{r}} d\alpha\right)^{\frac{-1}{s}}$$

$$= \left(\int_0^1 \{(1-\alpha)t^{-r}+\alpha\}^{\frac{-s}{-r}} d\alpha\right)^{\frac{1}{-s}} = F_{-s,-r}(t),$$

$$F_{s,r}(t^{-1})^{-1} = \left(\int_0^1 \{(1-\alpha)+\alpha t^{-r}\}^{\frac{s}{r}} d\alpha\right)^{\frac{-1}{s}}$$

$$= \left(\int_0^1 \{(1-\alpha)+\alpha t^{-r}\}^{\frac{-s}{-r}} d\alpha\right)^{\frac{1}{-s}} = F_{-s,-r}(t).$$

□

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